gives

$$0\leq \ldots \leq A_{s,s-1} \leq A_{s,s} \geq A_{s,s+1} \geq \ldots \geq 0 \tag{11}$$

where $A_{s,s}=1$ for a unity gain filter properly centered on ω_s.[6] If the controlled modes are the n_c lowest modes, and if high-performance filters are used, then Eq. (11) suggests that A^c will, in effect, be banded and A^r will, in effect, be primarily null except for a populated region in its lower left-hand corner.

We note that Meirovitch and Oz[1,3] have proposed an alternative approach to modal response measurement which requires, in general, several motion sensors but no filtering.

Consequences of Inaccurate Modal Information

The decoupling effectiveness of control actuator apportionings, Eqs. (7) and (10), depends on the accuracy of modal matrix Φ^{ac}. If the true modal matrix differs from the matrix used to calculate C^{ac}, which will generally be the case in practice, then decoupling will be incomplete. Moreover, the effectiveness of Eq. (10) depends additionally on the accuracy of the natural frequencies used for computation of A.

Reference 6 includes a limited numerical study of the reduction in effectiveness of Eq. (7) due to inaccurate modal information. The modal matrix of a "model" structure was used in the calculation of actuator apportionings to be applied on a similar "actual" structure. The differences between the "model" and "actual" structures were designed to be representative of differences that often exist between a finite-element modal and the actual hardware being analyzed. For the cases studied, the imperfect actuator apportionings produce only slight reductions in control effectiveness relative to corresponding cases with perfect apportionings. The results suggest that it would take substantial errors in mode shape estimates to render modal-space control ineffective or to produce serious instability.

Concluding Remarks

The most important factor limiting the effectiveness of modal-space control is control spillover into the residual modes which occurs because there are fewer control actuators than modes. However, if residual mode excitation can be acceptably minimized by a judicious choice of controlled modes and actuators, then the effectiveness of modal-space control appears to be reasonably insensitive to inaccurate modal parameters and to observation spillover resulting from realistic signal filtering.

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Time-Varying Weights for Optimal Control with Inequality Constraints

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Nomenclature

a_p	= applied acceleration, ft/s^2
a_t	= target acceleration, ft/s^2
b	= weight given to energy term in performance index
b_0	= initial value of b
E	= total mean-squared control energy, g^2-s^2
J	= performance index with constant weight
J'	= performance index with time-varying weight
R_1	= component of measurement error variance, rad^2-s
R_2	= contribution to measurement error variance from glint, rad^2-s^3
t	= time
t_f	= intercept time
v	= cross-range velocity, ft/s
V_x	= along-range velocity, ft/s
w	= process noise, ft/s^3
y	= cross-range displacement, ft
ϵ	= measurement noise, rad-s$^{1/2}$
σ	= observable angle, rad
τ	= target maneuver time constant, s

Introduction

INEQUALITY constraints, such as engine thrust limits or maximum permissible deflections of aerodynamic control surfaces, are encountered quite often in control system design. When these constraints prohibit linear quadratic Gaussian (LQG) formulations with Riccati equation solutions, the following characteristics of linear optimization solutions are sacrificed: 1) determination of optimal time-varying Kalman filter and closed-loop feedback gains, uniquely expressible in terms of the same type of formulation used for the optimal (Kalman) estimator; and 2) straightforward characterization for statistical performance of both estimator and controller.

In many applications, especially those involving stochastic processes, motivation for recovering these benefits is understandably great. A common approach is to omit the inequality constraint from the formulation while including, in the performance index to be minimized, a time integral of the constrained variable weighted by a constant. The resulting linear optimization is then repeatedly performed with different weightings, until that variable (or, for stochastic problems, a chosen multiple of its rms value) instantaneously reaches its maximum or minimum permissible level at some point in the solution. The performance index in that case is optimized without violating the constraint, but the time integral term in the optimization criterion does not provide a potentially beneficial lingering of the constrained variable near its extremal value, along an extremal arc, in the vicinity of the point just described. This prolonged hovering near an extremal value, generally associated with saturation nonlinearity, is achieved here using a linear formulation via time-varying weights.

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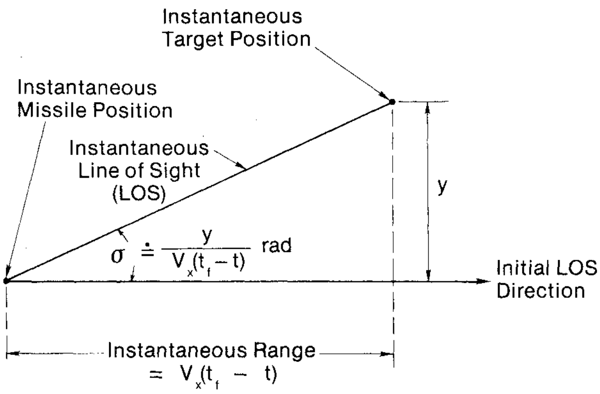

Fig. 1 Intercept geometry.

While optimal control theory has long included the use of time-varying weighting functions in the performance index, applications throughout the past two decades have almost always used time-invariant weights. This paper demonstrates potential benefits of time-variations in weighting, through a rudimentary illustrative example; the intent is to stimulate research in an area that appears to have been largely dormant thus far.

Original Formulation of an Intercept Application

The application used for illustration here is a planar intercept problem (Fig. 1), in which the control variable is applied missile acceleration in the y-direction.[1] As with many guidance problems, the requirements are somewhat conflicting (i.e., acceptably small rms terminal miss distance without exceeding 400 ft/s^2 applied acceleration). These objectives could in principle be pursued by disregarding optimization, and using the full allowable (400 ft/s^2) control acceleration magnitude at all times. In practice, however, there are at least two obvious objections to that policy:

1) Maximum control activity taken early in the engagement would be inefficient, since initial uncertainties in target dynamics would necessitate frequent reversals in maneuver direction.

2) Since induced drag increases with applied acceleration, inefficient control causes the missile's kinetic energy to be wasted.

Conventional missile guidance employs proportional navigation ("pronav") in which applied acceleration is proportional to the target/missile line-of- sight rate. Since that rate varies inversely with range, pronav reasonably allocates maneuver effort to scenario segments where maneuvering is most effective (although the terminal acceleration demands tend to be excessive). A better procedure, producing reasonable demands for control acceleration a_p near intercept, is to minimize the peformance index

$$J \triangleq \frac{1}{2}\left\langle [y(t_f)]^2 + b\int_0^{t_f} [a_p(t)]^2 \mathrm{d}t \right\rangle \qquad (1)$$

where t_f is the ratio of initial range (30,000 ft in this example) to V_x, and the angular brackets denote probabilistic ensembled averages; the weighting factor b is chosen for an acceptable compromise between terminal miss distance and the amount of control energy expended. The observable (σ in Fig. 1) was corrupted by additive white noise ϵ, with zero mean and variance of the form

$$\langle \epsilon^2 \rangle = R_1 + R_2/(t_f - t)^2 \qquad (2)$$

where $R_1 = 15\times(10)^{-6}$ rad^2-s and $R_2 = (1.67)^2 \times (10)^{-6}$ rad^2-s^3, corresponding to a glint level of 1.67 mrad at 3000 ft, for a speed V_x of 3000 ft/s.‡ Target acceleration a_t was characterized by exponential correlation with a 2-s time constant τ and an rms value of 100 ft/s^2, in the state dynamic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y \\ v \\ a_t \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1/\tau \end{bmatrix}\begin{bmatrix} y \\ v \\ a_t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} a_p + \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix} \qquad (3)$$

where w is zero mean white noise with a variance of 10^4 (ft/s^3)^2. Initial rms uncertainties were 200 ft/s in v and 100 ft/s^2 in a_t.

When b was set to the constant 0.0151 the time history shown by the dotted curve of Fig. 2 was obtained. This value was chosen so that rms control acceleration never exceeds 400 ft/s^2. Total mean squared control

$$E = \frac{1}{2}\left\langle \int_0^{t_f}\left(\frac{a_p(t)}{32.174}\right)^2 \mathrm{d}t \right\rangle \qquad (4)$$

and the rms miss $\langle [y(t_f)]^2 \rangle^{1/2}$ were approximately 165 (g-s)^2 and 32 ft, respectively.

Reformulation with Generalized Weighting

Suppose that it became necessary to reduce the rms miss distance below 32 ft, without changing the maximum instantaneous rms acceleration or any other system parameter. This requirement can be met, *without sacrificing the convenient Riccati formulation,* by using the more general performance index

$$J' = \frac{1}{2}\left\langle [y(t_f)]^2 + \int_0^{t_f} b(t)[a_p(t)]^2 \mathrm{d}t \right\rangle \qquad (5)$$

with a time varying weighting $b(t)$. In this way, the constraint (maximum allowable acceleration in this case) can be reached and maintained over a finite time interval.

A simple iterative algorithm was devised to synthesize the weighting function, $b(t)$, which subsequently provided a solution with a control acceleration a_p reaching a constant magnitude limit over a finite time interval (also shown in Fig. 2). Although a_p appears to be a nonlinear (hard limited) function, it is realized with time-varying gains in a linear formulation. The limited space of this Note does not permit a full description of the iterative algorithm developed to obtain the weighting function, but it is based on an extension of the system parameter identification approaches discussed in Refs. 2-4, augmented by an adaptive gain adjustment feature to speed convergence. Irrespective of this algorithm, the point being emphasized here is that the weighting function shown in Fig. 2 achieves the objectives defined by Eq. (5) and the constraints defined in the accompanying discussion.

Some further interpretive observations are in order here. First, with a weighting constant b_0 fixed at its initial value in Fig. 2 over the time interval t_f, the unconstrained acceleration exceeds the 400 ft/s^2 level for about 0.5 s as shown. The constrained acceleration corresponding to the variable weighting function shown, however, adheres to the 400 ft/s^2 level for a longer duration (about 0.8 s), resulting in an energy increase ΔE of less than 13% (from about 165.5 to 186.5 g^2-s^2) for a 20% reduction in rms miss distance Δy (from about 32 to 25.6 ft). Additional optimal solutions with time-varying weighting were obtained for different values of initial weight b_0. As this initial value was reduced, increased per-

‡There is a misprint in Ref. 1, in that the numerical value given for R_2 was the square root of the value actually used.

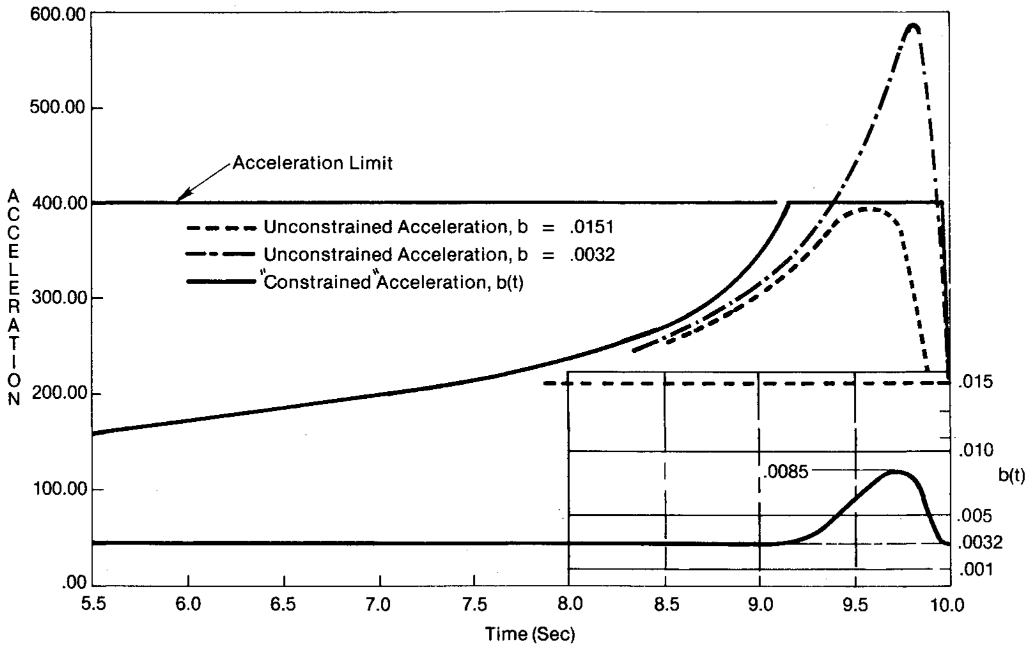

Fig 2 Control acceleration obtained from constant and time-varying weighting functions.

centages of energy were incurred to achieve further reductions in rms miss distance. These percentage increases in total expended energy, however, are in many cases acceptable (e.g., much more readily available than an increased constraint level for given aerodynamic limits). Further solutions were obtained at several values of the constraint levels, with qualitatively similar results.

Conclusions

While optimal controls extremalize only a given scalar performance index, researchers have been understandably reluctant to abandon the formalism in the presence of inequality constraints which induce an additional quadratic optimality criterion with a constant weight. The performance index may then inhibit a preferred "full throttle" operation over a time interval. In applications allowing obvious physical interpretations, for example, maximum allowable control effort could be prolonged to reduce a terminal state deviation; to do so, however, would disallow using the convenient Riccati equation formalism in most linear optimization applications published to date. Replacement of constant weights by time-varying weighting functions, surprisingly rare in applications literature, is advocated here to permit usage of the full linear formalism while keeping the performance index selections more consistent with the true intent of the optimization. The rudimentary example and approach to synthesizing a time varying weighting function presented here are intended to stimulate further development for broader application.

Acknowledgment

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Identity Between INS Position and Velocity Error Models

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Nomenclature

A,B,C	= general vectors
f	= specific force vector
g	= gravity vector
g^m	= gravitation vector
Δg	= error in computed g vector due to error in assumed position
δg	= gravity deflection and anomaly vector
R	= position vector
ΔR	= position error vector
V	= ground velocity vector
ΔV	= INS-computed velocity error vector
ρ	= angular rotation rate vector of the t frame with respect to the e frame (see Ref. 1)
ψ	= vector angle by which a rotation of the c frame ends at the p frame (see Ref. 1)
Ω	= Earth rate vector
ω	= angular rotation rate vector of the t frame with respect to an inertial frame (see Ref. 1)
∇	= accelerometer error vector
$\dot{A}_q$	= rate of change of the general vector A relative to a general coordinate system q

Subscripts

e	= Earth-fixed coordinate system (see Ref. 1)
i	= inertial coordinate system
q	= general coordinate frame
t	= true coordinate system (see Ref. 1)

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